Parameter Estimation and Tracking Control of MIMO Linear Systems Without Prior Knowledge of Control Signs and Parameter Bounds

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***Abstract*— Dealing with the uncertain high-frequency gain matrix, denoted as K_p, is a fundamental problem in multivariable adaptive control systems. In this paper, we propose a new solution for parameter estimation and adaptive control for a general class of multi-input multi-output discrete-time linear time-invariant systems. The proposed scheme does not require any prior knowledge of the sign or bound information of K_p, and thus, significantly relaxes the design conditions in traditional multivariable adaptive control systems. Compared with the commonly used Nussbaum gain or multi-model techniques for addressing the unknown signs of K_p, the proposed scheme does not rely on any additional design conditions or any switching mechanism, while still ensuring closed-loop stability and asymptotic output tracking. Specifically, an output feedback adaptive control law is developed based on a matrix decomposition technique, which leads to derivation of a modified estimation error model. Subsequently, a gradient-based parameter update law is formulated only relying on the non-zero condition of the leading principle minors of K_p. Through designing gain functions and stable filters, the controller is always non-singular and does not involve any causal contradiction problem. Simulation study showcases the design process and demonstrates the effectiveness of the proposed scheme.**

***Index Terms*— Parameter estimation, asymptotic output tracking, high-frequency gain matrix, singularity problem**

I. INTRODUCTION

THE presence of uncertainties within systems and environments poses a significant challenge to the feasibility of control algorithms and the overall system performance. Adaptive control, serving as a powerful tool for effectively addressing parametric, structural, and environmental uncertainties, has gained popularity across various engineering and scientific domains. Over the past few decades, substantial efforts have been dedicated to the field of adaptive control, aiming to counteract the adverse effects of uncertainties. These endeavors have resulted in numerous notable achievements, as evidenced by a lot of works (see, for example, [1]–[8]).

It is common for practical systems to have multiple inputs and multiple outputs, especially in emerging technologies. As a result, the adaptive control of multi-input multi-output (MIMO) systems has been a focal point for control researchers, given its theoretical challenges and practical significance. In contrast to single-input single-output (SISO) systems, multivariable systems feature distinct characteristics such as dynamic coupling between input and output signals, a unique structure referred to as the interactor matrix, and a specific gain matrix known as the high-frequency gain matrix. These characteristics give rise to novel challenges for the adaptive control design and analysis of MIMO systems. In recent decades, research in multivariable adaptive control has addressed a multitude of important and challenging problems, yielding a great deal of rigorous and promising solutions (see, for example, [9]–[14]). It is worth mentioning that the renowned backstepping technique, initially introduced in [3], plays a significant role in the adaptive control design and analysis of high-order nonlinear systems.

In spite of remarkable success, numerous open issues persist in the realm of adaptive control for MIMO systems and merit further study and exploration. A fundamental and longstanding challenge lies in addressing the uncertainty associated with the high-frequency gain matrix, denoted as K_p. In some earlier results, it is assumed that a matrix S_p is known, satisfying the condition that K_pS_p is symmetric positive ([1], [15], [16]). Leveraging this assumption, a stable parameter estimator can be formulated to achieve parameter estimation and simultaneously avoid singularity of the adaptive control law. In [17], an improved adaptive control method was devised under a less restrictive condition $K_pS_p^T + S_pK_p^T > 0$ with a known matrix S_p for systems with a fixed vector relative degree. Subsequently, employing some matrix decomposition techniques, various model reference adaptive control (MRAC) methods have been proposed under the assumption that the leading principal minors of K_p are non-zero and their signs are known ([15], [18], [19]). This assumption parallels the

This work was supported in part by the National Natural Science Foundation of China under Grants 62173323, 62322304, 62433020 and T2293770, in part by the Beijing Institute of Technology Research Fund Program for Young Scholars, and in part by the Foundation under Grant 2019-JCJQ-ZD-049. *Corresponding author:* Yanjun Zhang.

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prior knowledge of sign information about the high-frequency gain, denoted as k_p, in SISO systems, and substantially eases the conditions imposed on K_p.

Despite considerable progress, the requirement for prior sign information still restricts the applicability of adaptive control methods. The sign information of the high-frequency gain (matrix) indicates the control directions and may be unknown for many practical control problems in engineering, such as autopilot design of uncertain ships and uncalibrated visual servoing ([20]). Hence, further relaxing the conditions on the high-frequency gain (matrix) in adaptive control is of paramount importance. For the case of SISO systems, achievements on this issue are remarkable. A pioneering work is [21], where the author proposed a controller by introducing a function, known as Nussbaum gain function, to stabilize a special class of SISO continuous-time first-order systems without requiring the sign information of k_p. Since then, Nussbaum gain function based control methods have been extensively investigated for various SISO systems ([22]–[29]). Recently, [30] first extended the Nussbaum gain function technique to MIMO systems to cope with unknown sign knowledge of K_p required in [18] and [19], where a novel multiple Nussbaum gain functions and backstepping based adaptive control method was developed for a class of MIMO continuous-time linear time-invariant (LTI) systems. Besides Nussbaum gain function, the multi-model switching technique is also used in adaptive control systems to tackle the uncertain sign information ([31]–[34]). This kind of technique employs multiple controllers and estimators to seek for appropriate control directions by introducing a switching mechanism. Taking [31] as an example, a projection based state feedback multiple-model adaptive control method was designed for MIMO continuous-time systems without requiring information about the signs of K_p, but assuming the prior bound knowledge of unknown parameters, where 2^M controllers and 2^M parameter estimators were deployed for a system with M inputs and M outputs. In addition, some other methods have been reported in the literature to address the challenge of unknown sign information of K_p. For instance, an indirect multivariable adaptive control method was developed in [35] to weaken the assumptions of high-frequency gain matrix, where a hysteresis and projection mechanism was used to avoid singularity and an upper bound on the norm of K_p was needed as the prior knowledge. A modified model reference adaptive controller for multivariable continuous-time systems was introduced in [36], [37] by using the dynamic regressor extension and mixing parameter estimation technique, which removed the necessity for prior knowledge of K_p but required an additional interval excitation condition on the regressor vector.

Although some advancements have been made for dealing with unknown sign information of K_p in adaptive control of MIMO systems, some open problems still persist. As highlighted in [38], [39], the Nussbaum gain method is known to result in adverse oscillation performance due to the nature of Nussbaum gain function. Regarding the multi-model switching scheme, as indicated in [30], it usually incurs a substantial computational burden and needs prior bound knowledge of uncertain parameters. Moreover, persistent controller switching may exist in the multi-model adaptive control. In addition, the majority of works about this problem tend to focus on systems in continuous-time with discrete-time systems receiving limited attention. Recently, [40] proposed a novel MRAC method for a special class of SISO continuous-time systems with relative degree one. This method removes the need for prior information of k_p and achieves asymptotic tracking performance without resorting to Nussbaum gain function or multi-model technique. An extension to SISO discrete-time systems with arbitrary relative degrees is presented in [41]. However, the applicability of the method proposed in [40] and [41] to MIMO scenarios for dealing with unknown sign and bound information of the high-frequency gain matrix K_p remains unclear.

Motivated by the aforementioned observations, two questions may be raised: "what level of knowledge regarding K_p is required for adaptive control?" and "how can the adverse persistent switching issue often occuring in existing results be avoided". In this paper, we give analytical solutions to the above questions and propose a novel modified MRAC scheme for a general class of MIMO discrete-time LTI systems. The main contributions of this paper are as follows.

(i) A new matrix decomposition based output feedback adaptive controller along with a gradient based parameter update law is formulated for a general class of MIMO discrete-time LTI systems. It ensures closed-loop stability and asymptotic output tracking without introducing any switching mechanism that often used in existing methods, such as Nussbaum gain function based method, multi-model based method, etc.

(ii) In comparison to the existing results, the proposed adaptive control scheme removes the need for prior sign information of K_p, obviates the requirement for knowledge of system uncertain parameter bounds, and does not rely on any type of excitation condition. Regarding the question of "what level of knowledge regarding K_p is required for adaptive control", our solution demonstrates that only the non-zero information pertaining to the leading principal minors of K_p is needed.

(iii) The developed adaptive controller and parameter update law are guaranteed to remain non-singular at all times by incorporating well-designed time-varying gain functions. The introduction of some filtered operators to handle unavailable signals in the adaptive control process guarantees that the proposed control method does not involve any casuality contradiction problem.

The rest of this paper is organized as follows. In section II, we articulate the problem to be addressed and the technical issues to be addressed. Section III comprehensively presents the entire adaptive control design process, encompassing the controller structure, the form of parameter update law, and the stability analysis and tracking performance analysis of the closed-loop system. The simulation study is depicted in Section IV. Finally, Section V presents the concluding remarks.

Notation. Throughout this paper, $\mathbb{R}$ denotes the set of real numbers. The symbols z and z^{-1} represent the time advance and time delay operators, respectively, i.e., $z[x](t) = x(t+1)$

and $z^{-1}[x](t) = x(t-1)$, where $t \in \{0, 1, 2, 3, ...\}$, $x(t) \triangleq x(tT)$ for a sampling period $T > 0$, and $x(t)$ denotes any signal of any finite dimension. We employ L^∞ and L^2 to denote signal spaces defined as $L^\infty = \{x(t) : ||x(\cdot)||_\infty < \infty\}$ and $L^2 = \{x(t) : ||x(\cdot)||_2 < \infty\}$ with $||x(\cdot)||_\infty = \sup_{t\geq 0} \max_{1\leq i\leq n} |x_i(t)|$ and $||x(\cdot)||_2 = (\sum_{t=0}^{\infty} |x_1(t)|^2 + \cdots + |x_n(t)|^2)^{\frac{1}{2}}$, where $x(t) = [x_1(t), ..., x_n(t)]^T$ denotes any signal on $\mathbb{R}^n$. We use $c > 0$ to denote a generic signal bound, and $\tau(t)$ to denote a generic function belonging to $L^2 \cap L^\infty$ which converges to zero as t tends to ∞. For a vector signal $x(t)$, $\|x(t)\|$ denotes its Euclidean norm. The notation $\text{diag}\{b_1, ..., b_N\}$ represents the diagonal matrix with diagonal elements $b_1, ..., b_N$. The notation I_M denotes the identity matrix of size $M \times M (M \geq 2)$, while 0_M denotes a square matrix of size $M \times M$ with all elements equal to zero.

II. Problem statement

This section presents the formulation of the system model, the control objective, and design conditions. Additionally, it also outlines the technical issues to be resolved in this paper.

A. System Model

Consider the following MIMO discrete-time LTI system model

$$y(t) = G(z)[u](t), \ t \geq t_0, \tag{1}$$

where $y(t) = [y_1(t), ..., y_M(t)] \in \mathbb{R}^M$ is the system output vector and $u(t) = [u_1(t), ..., u_M(t)] \in \mathbb{R}^M$ is the system input vector. The transfer matrix $G(z) = Z(z)P^{-1}(z)$ is an $M \times M$ strictly proper rational matrix with full rank, where $Z(z)$ and $P(z)$ are $M \times M$ right coprime polynomial matrices. The matrix $P(z)$ is both column proper and possesses a degree equal to the controllability index of $G(z)$. In this work, we particularly focus on systems with multiple inputs and multiple outputs, i.e., $M \geq 2$. This facilitates consistent expressions of some definitions in the following. For the case of $M = 1$, a valid solution can be found in [41].

For the adaptive control of the system (1), we initially introduce the following lemma. This lemma elucidates a crucial concept concerning MIMO discrete-time LTI systems, referred to as the modified left interactor matrix, which plays a significant role in parametrization and adaptive control design.

Lemma 1: ([15]) *For any $M \times M$ strictly proper full rank rational matrix $G(z)$, there exists an $M \times M$ lower triangular polynomial matrix $\xi_m(z)$, of the form*

$$\xi_m(z) = \begin{bmatrix} d_1(z) & 0 & \cdots & 0 & 0 \\ h_{21}^m(z) & d_2(z) & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h_{M1}^m(z) & \cdots & \cdots & h_{MM-1}^m(z) & d_m(z) \end{bmatrix},$$

where $h_{ij}^m(z)$, $j = 1, ..., M-1$, $i = 2, ..., M$, are polynomials and $d_i(z)$, $i = 1, ..., M$, are monic stable polynomials of degree $l_i > 0$, such that $K_p \triangleq \lim_{z\to\infty} \xi_m(z)G(z)$ is finite and nonsingular.

The proof of this lemma is available in [15]. The matrix $\xi_m(z)$ is called the modified left interactor matrix of $G(z)$. It possesses a proper and stable inverse, providing characterization of the zero structure at infinity of $G(z)$. Meanwhile, K_p is referred to as the high-frequency gain matrix of $G(z)$.

B. Control Objective and Design Conditions

Control objective. The control objective of this paper is to design an output feedback adaptive control input $u(t)$ for the system (1) with unknown $G(z)$ such that the closed-loop system is stable and the output $y(t)$ tracks a given bounded reference output signal $y^*(t) \in \mathbb{R}^M$ asymptotically, where

$$y^*(t) = W_m(z)[r](t) \tag{2}$$

with $W_m(z)$ being an $M \times M$ rational transfer matrix and $r(t) \in \mathbb{R}^M$ being a bounded external reference input signal.

Assumptions. To accomplish the control objective, the following design conditions are needed.

(A1). All zeros of $G(z)$ are stable.
(A2). The observability index v of $G(z)$ is known.
(A3). $G(z)$ has a known modified left interactor matrix $\xi_m(z)$, all poles of $W_m(z)$ are stable, and the zero structure at infinity of $W_m(z)$ is the same as that of $G(z)$, i.e., $\lim_{z\to\infty} \xi_m(z)W_m(z)$ is finite and nonsingular.
(A4). All leading principal minors of the high-frequency gain matrix K_p, denoted as $\Delta_1, \Delta_2, ..., \Delta_M$, are non-zero.

Assumption (A1) implies the minimum phase nature of the system (1). and is a standard design condition in the literature ([2], [10], [15], [16]). Assumption (A2) is employed to determine the dimension of estimated parameter vectors and can be further relaxed as an upper bound on the observability index v of $G(z)$ is known. Assumption (A3) is analogous to the relative degree condition of a SISO system, and it is essential for selecting a stable reference model system that generates $y^*(t)$ for model matching. According to Assumption (A3), without loss of generality, we choose $W_m(z) = \xi_m^{-1}(z)$ for the reference model (2), where $\xi_m^{-1}(q_m z)$ is stable for some $q_m \in (0, q) \subset (0, 1)$ ([2], [15], [16]). It is noted that $\xi_m(z)$ may depend on the parameters of $G(z)$. However, some techniques can make $\xi_m(z)$ diagonal, allowing it to be specified independently of $G(z)$ ([15]). In this paper, we mainly focus on addressing the uncertainty associated with the high-frequency gain matrix K_p. Thus, we assume $\xi_m(z)$ is known a priori. Assumption (A4) is akin to assuming that the high-frequency gain of SISO systems is non-zero. We illustrate Assumption (A4) by using a linearized aircraft control system in [42]. Reference [42] shows that when the inputs consists of engine throttle, elevator, and rudder, and the outputs are forward velocity, pitch angle, and yaw angle, the high-frequency gain matrix takes the form of $K_p = [k_{p1}, k_{p2}, k_{p3}] \in \mathbb{R}^{3\times 3}$, where $k_{p1} = [b_{011}, b_{031}, 0]^T, k_{p2} = [b_{012}, b_{032}, 0]^T, k_{p3} = [0, 0, b_{064}\cos(1/\theta_0)]^T$. As elucidated in [42], b_{011}, b_{012}, b_{031}, b_{032}, and b_{064} are the control gains from engine throttle to forward acceleration, from the elevator to forward acceleration, from engine throttle to pitch acceleration, from the elevator to pitch acceleration, and from rudder to yaw acceleration, respectively. Moreover, θ_0 represents the value of the Euler pitch angle θ at the wings-level steady-state equilibrium point that ensures $\cos(1/\theta_0) > 0$. Consequently, in light of the structure of K_p, Assumption (A4) necessitates that $\Delta_1 = b_{011} \neq 0$, $\Delta_2 = b_{011}b_{032} - b_{012}b_{031} \neq 0$, and $\Delta_3 = (b_{011}b_{032} - b_{012}b_{031})b_{064} \neq 0$. This example illustrates the reasonability of Assumption (A4). If some principal minors of K_p are zero but K_p is nonsingular, a permutation matrix P can be introduced such that PK_p possesses nonzero leading principal minors. With this modification, the control design can proceed using the modified high-frequency gain matrix PK_p ([15]).

C. Clarification of High-Frequency Gain Matrix Issue

In discrete-time adaptive control for the system (1), traditional assumptions are (A1)-(A3), supplemented by an additional requirement on K_p as

(A4a). An $M \times M$ matrix S_p is known such that $2I_M > K_pS_p = (K_pS_p)^T > 0$.

Evidently, this assumption is much more restrictive than Assumption (A4). This is due to the fact that the traditional model reference adaptive controlller results in a bilinear form error model. In such a scenario, (A4a) is necessary to derive a stable parameter update law. Readers are referred to [15] for further details about (A4a) based MRAC design. In the SISO case, that is $M = 1$, K_p reduces to a scalar whose sign information is sufficient for devising a stable parameter update law, i.e., S_p can be selected as the sign of k_p. However, in the MIMO scenario, determining a prior knowledge of K_p for a stable parameter update law proves considerably challenging, i.e., S_p in Assumption (A4a) is hard to choose. Thus, (A4a) indicates a certain level of restrictiveness when applying traditional MRAC in practical situations.

In fact, the condition (A4a) on K_p can be relaxed if certain structural information about K_p is accessible. One of those is as follows.

(A4b). All leading principal minors of the high-frequency gain matrix K_p are non-zero and their signs are known. Some upper bounds d_i^0 on $|d_i^*|$, i.e., $|d_i^*| < d_i^0$, $i = 1, ..., M$, are known with $d_1^* \triangleq \Delta_1, d_i^* \triangleq \frac{\Delta_i}{\Delta_{i-1}}, i = 2, ..., M$.

The signs of the leading principal minors of K_p in Assumption (A4b) are commonly referred to as the sign information of K_p. This information reflects control directions in practical applications and can be acquired at times by considering the physical meanings of inherent system characteristics. Therefore, compared with Assumption (A4a) based MRAC, control methods based on (A4b) impose relatively less restrictive conditions on K_p, rendering them more practical for real-world applications. Currently, assuming the known sign information of K_p to derive stable parameter estimators remains the predominant method in the field of adaptive control.

However, addressing the unknown sign information of the high-frequency gain (matrix) persists as a fundamental and enduring challenge in adaptive control, particularly within the domain of multivariable adaptive control. Actually, the sign information of high-frequency gain (matrix) may often be unknown in various engineering control problems ([20]). Although big progress has been made in adaptive control for SISO systems with unknown sign information of the high-frequency gain, there is a notable scarcity of studies addressing the design of control methods for MIMO systems without knowledge of the sign information of K_p. It is noteworthy that most adaptive control methods employ the Nussbaum gain function to address unknown control directions, which is initially developed for SISO systems and extended a lot till now ([21]). In [30], a Nussbaum gain function based adaptive backstepping control method is proposed for MIMO continuous-time LTI systems, which only requires the non-zero condition of leading principal minors of K_p. However, the proposed scheme may encounter a system oscillation problem. This is attributed to the inclusion of oscillation functions, representing a typical problem associated with Nussbaum gain function methods. Regarding the multi-model switching method, the inherent disadvantages are that the prior bound knowledge of the uncertain parameters needs to be known, parameter estimation for multiple models often leads to computational burden, and controller switching may be persistent. Consequently, there is still an open challenge in designing control strategies for MIMO systems that can overcome the limitations imposed by unknown control directions without resorting to Nussbaum gain function or multi-model switching. Particularly, even less work has been done on discrete-time MIMO systems.

D. Technical Issues

In contrast to traditional multivariable adaptive control methods, this study solely relies on the assumption that the leading principal minors of K_p are all non-zero, as stated in Assumption (A4). It is obvious that (A4) demands less information on K_p compared witn (A4b). Consequently, traditional design methods do not work under this setting.

Recently, [40] proposed a new control method to address unknown control directions for SISO continuous-time LTI systems with relative degree one. An extension to SISO discrete-time LTI systems with arbitrary relative degrees can be found in [41]. In this study, we aim to extend the novel method proposed in [40] and [41] to control MIMO discrete-time LTI systems (1), removing the need for prior sign information and bound knowledge of K_p. It is far from trivial due to some special challenges associated with multivariable adaptive control. Overall, we will address the following challenging technical issues in this paper

- how to handle dynamic coupling between the inputs and the outputs to seek for a suitable parameterized model for parameter estimation and control design of the system (1);
- how to design a well-defined control input vector $u(t)$ to prevent the introduction of sign or bound information of K_p into parameter update law for the system (1);
- how to circumvent control gain singularity problems during the adaptive process and ensure a reasonable casuality in the adaptive control of the system (1); and
- how to conduct stability analysis and analyze the tracking performance of the closed-loop system.

III. A NEW SOLUTION TO MULTIVARIABLE MRAC

In this section, we elaborate on a novel adaptive control design method. We begin by introducing a fundamental design equation for model reference control of MIMO discrete-time LTI systems. Subsequently, we present a decomposition form of K_p which enables the development of a suitable parameterized model for controller design. Following this, we present the form of the developed adaptive control law. Further, a gradient based parameter update law is developed by use of a new type of estimation error model. Finally, we give the main result of this study, where the stability and tracking performance analysis is conducted to substantiate the effectiveness of the control design.

A. Design Equation

First, we present a fundamental design equation vital for multivariable MRAC and essential for the subsequent adaptive control design.

Define the following signals

$$\omega_1(t) = F(z)[u](t), \omega_2(t) = F(z)[y](t),$$
$$F(z) = \frac{A(z)}{\Lambda(z)}, A(z) = \left[I_M, zI_M, \cdots, z^{v-2}I_M\right]^T, \quad (3)$$

where $\Lambda(z)$ is a monic polynomial of degree $v-1$ such that $\Lambda(q_0 z)$ is stable for some $q_0 \in (0,1)$. Then, with the above specifications of $\Lambda(z), \xi_m(z), P(z), Z(z)$, we have the following lemma.

Lemma 2: ([15]) *There exist some constant parameter matrices* $\Theta_1^* = \left[\Theta_{11}^*, \cdots, \Theta_{1v-1}^*\right]^T$, $\Theta_2^* = \left[\Theta_{21}^*, \cdots, \Theta_{2v-1}^*\right]^T$ *with* $\Theta_{ij}^* \in \mathbb{R}^{M\times M}$, $i=1,2,\ j=1,...,v-1$, $\Theta_{20}^* \in \mathbb{R}^{M\times M}$, *and* $\Theta_3^* = K_p^{-1} \in \mathbb{R}^{M\times M}$ *such that*

$$\Theta_1^{*T}A(z)P(z) + (\Theta_2^{*T}A(z) + \Theta_{20}^*\Lambda(z))Z(z)$$
$$= \Lambda(z)\left(P(z) - \Theta_3^*\xi_m(z)Z(z)\right). \quad (4)$$

The proof of this lemma can be found in [15]. Actually, the equation (4) is the well-known matching equation for output feedback model reference control of MIMO discrete-time systems.

From (4) and the system model (1), we get

$$I_M - \Theta_1^{*T}F(z) - \Theta_2^{*T}F(z)G(z) - \Theta_{20}^*G(z)$$
$$= \Theta_3^*\xi_m(z)G(z). \quad (5)$$

Operating both sides of (5) by any control input $u(t) \in \mathbb{R}^M$, we derive the system signal identity equation

$$u(t) - \Theta_1^{*T}\omega_1(t) - \Theta_2^{*T}\omega_2(t) - \Theta_{20}^*y(t)$$
$$= K_p^{-1}\xi_m(z)[y](t). \quad (6)$$

Combining the equation (6) and the reference model (2), we obtain

$$K_p\left(u(t) - \Theta_1^{*T}\omega_1(t) - \Theta_2^{*T}\omega_2(t) - \Theta_{20}^*y(t) - \Theta_3^*r(t)\right)$$
$$= \xi_m(z)\left[y - y^*\right](t). \quad (7)$$

Now we get a parameterized model (7) for the system (1) and the reference model (2).

Remark 1: When the parameters in the system (1) were entirely known, the parameterized model (7) inspires the formulation of a control law as

$$u(t) = \Theta_1^{*T}\omega_1(t) + \Theta_2^{*T}\omega_2(t) + \Theta_{20}^*y(t) + \Theta_3^*r(t),$$

which would achieve accurate output tracking for any given reference output $y^*(t)$. Naturally, an adaptive control law can be designed as

$$u(t) = \Theta_1^T(t)\omega_1(t) + \Theta_2^T(t)\omega_2(t) + \Theta_{20}(t)y(t)$$
$$+\Theta_3(t)r(t) \quad (8)$$

with $\Theta_1(t)$, $\Theta_2(t)$, $\Theta_{20}(t)$, and $\Theta_3(t)$ denoting estimates of Θ_1^*, Θ_2^*, Θ_{20}^*, and Θ_3^*, respectively, and being updated by developed parameter update laws. The control law (8) is a traditional output feedback MRAC law ([15]). Regrettably, this adaptive control law requires (A4a) to establish a stable parameter update law as clarified in [15], which is restrictive in practical applications.

B. Decomposition of K_p

Because the control law (8) induced by the parameterized model (7) cannot achieve the control objective under Assumption (A4), we need to derive a different parameterized model suitable for the following control design. Based on Assumption (A4), K_p can be decomposed as

$$K_p = LD^*U, \quad (9)$$

where L is an $M\times M$ unit lower triangular matrix, U is an $M\times M$ unit upper triangular matrix, and

$$D^* = \text{diag}\left\{d_1^*, d_2^*, ..., d_M^*\right\}$$

with $d_1^* \triangleq \Delta_1$, $d_i^* \triangleq \frac{\Delta_i}{\Delta_{i-1}}$, $i = 2, ..., M$. Utilizing the decomposition equation (9), we express (7) as

$$D^*U\left(u(t) - \Theta_1^{*T}\omega_1(t) - \Theta_2^{*T}\omega_2(t) - \Theta_{20}^*y(t) - \Theta_3^*r(t)\right)$$
$$= L^{-1}\xi_m(z)\left[y - y^*\right](t).$$

Further, we have

$$D^*\left(u(t) - \Phi_0^*u(t) - \Phi_1^{*T}\omega_1(t) - \Phi_2^{*T}\omega_2(t) - \Phi_{20}^*y(t) - \Phi_3^*r(t)\right)$$
$$= \xi_m(z)[y - y^*](t) + \Theta_0^*\xi_m(z)[y - y^*](t), \quad (10)$$

where $\Phi_{20}^* \triangleq U\Theta_{20}^*$, $\Phi_i^* \triangleq \Theta_i^*U^T, i = 1,2$, $\Phi_3^* \triangleq U\Theta_3^*$, $\Phi_0^* \triangleq I_M - U$ possesses the triangular form as

$$\Phi_0^* = \begin{bmatrix} 0 & \phi_{12}^* & \phi_{13}^* & \cdots & \phi_{1M}^* \\ 0 & 0 & \phi_{23}^* & \cdots & \phi_{2M}^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \phi_{M-1M}^* \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}, \quad (11)$$

and $\Theta_0^* \triangleq L^{-1} - I_M$ possesses the triangular form as

$$\Theta_0^* = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \theta_{21}^* & 0 & 0 & \cdots & 0 \\ \theta_{31}^* & \theta_{32}^* & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \theta_{M-11}^* & \cdots & \theta_{M-1M-2}^* & 0 & 0 \\ \theta_{M1}^* & \cdots & \theta_{MM-2}^* & \theta_{MM-1}^* & 0 \end{bmatrix}. \quad (12)$$

Define the tracking error as $e(t) = y(t) - y^*(t)$. Then, we denote

$$\begin{aligned}
&\omega(t) = \left[\omega_1^T(t), \omega_2^T(t), y^T(t), r^T(t)\right]^T, \\
&e_\xi(t) = \xi_m(z)\,[e]\,(t) = [e_{\xi 1}(t), \cdots, e_{\xi M}(t)]^T, \\
&e_{\theta i}(t) = [e_{\xi 1}(t), \cdots, e_{\xi i-1}(t)]^T \in \mathbb{R}^{i-1},\ i = 2, ..., M, \quad (13)
\end{aligned}$$

and

$$\begin{aligned}
\chi_1(t) &= \left[u_2(t), \cdots, u_M(t), \omega^T(t)\right]^T, \\
\chi_2(t) &= \left[u_3(t), \cdots, u_M(t), \omega^T(t)\right]^T, \\
&\vdots \\
\chi_{M-1}(t) &= \left[u_M(t), \omega^T(t)\right]^T, \\
\chi_M(t) &= \omega(t). \quad (14)
\end{aligned}$$

For Φ_0^* in the form of (11), we denote

$$\begin{aligned}
\phi_1^* &= \left[\phi_{12}^*, \phi_{13}^*, \cdots, \phi_{1M}^*, \left[\Phi_1^{*T}, \Phi_2^{*T}, \Phi_{20}^*, \Phi_3^*\right]_1\right]^T, \\
\phi_2^* &= \left[\phi_{23}^*, \phi_{24}^*, \cdots, \phi_{2M}^*, \left[\Phi_1^{*T}, \Phi_2^{*T}, \Phi_{20}^*, \Phi_3^*\right]_2\right]^T, \\
&\vdots \\
\phi_{M-1}^* &= \left[\phi_{M-1M}^*, \left[\Phi_1^{*T}, \Phi_2^{*T}, \Phi_{20}^*, \Phi_3^*\right]_{M-1}\right]^T, \\
\phi_M^* &= \left[\Phi_1^{*T}, \Phi_2^{*T}, \Phi_{20}^*, \Phi_3^*\right]_M^T,
\end{aligned}$$

where $[\Phi_1^{*T}, \Phi_2^{*T}, \Phi_{20}^*, \Phi_3^*]_i$ is the i-th row of

$$[\Phi_1^{*T}, \Phi_2^{*T}, \Phi_{20}^*, \Phi_3^*].$$

For Θ_0^* in the form of (12), we further denote

$$\begin{aligned}
\theta_2^* &= \theta_{21}^* \in \mathbb{R}, \\
\theta_3^* &= [\theta_{31}^*, \theta_{32}^*]^T \in \mathbb{R}^2, \\
&\vdots \\
\theta_{M-1}^* &= [\theta_{M-11}^*, \cdots, \theta_{M-1M-2}^*]^T \in \mathbb{R}^{M-2}, \\
\theta_M^* &= [\theta_{M1}^*, \cdots, \theta_{MM-1}^*]^T \in \mathbb{R}^{M-1}.
\end{aligned}$$

Then, the equation (10) can be expressed as

$$\begin{aligned}
&e_{\xi 1}(t) = d_1^*(u_1(t) - \phi_1^{*T}\chi_1(t)), \\
&e_{\xi i}(t) + \theta_i^{*T} e_{\theta i}(t) = d_i^*(u_i(t) - \phi_i^{*T}\chi_i(t)), i = 2, ..., M, \quad (15)
\end{aligned}$$

for any control input $u(t) = [u_1(t), ..., u_M(t)]^T$. Now we get K_p decomposition based parameterized models (10) and (15), which are crucial for the following control design.

Remark 2: The K_p decomposition based parameterized model (10) indicates that a control law designed as

$$\begin{aligned}
u(t) = \Phi_0^* u(t) + \Phi_1^{*T}\omega_1(t) + \Phi_2^{*T}\omega_2(t) + \Phi_{20}^* y(t) \\
+ \Phi_3^* r(t)
\end{aligned}$$

would achieve accurate output tracking if the parameters of the system (1) were known. Accordingly, an adaptive control law can be designed as

$$\begin{aligned}
u(t) = \Phi_0(t)u(t) + \Phi_1^T(t)\omega_1(t) + \Phi_2^T(t)\omega_2(t) + \Phi_{20}(t)y(t) \\
+ \Phi_3(t)r(t), \quad (16)
\end{aligned}$$

where $\Phi_0(t),\ \Phi_1(t),\ \Phi_2(t),\ \Phi_{20}(t)$, and $\Phi_3(t)$ denote estimates of $\Phi_0^*,\ \Phi_1^*,\ \Phi_2^*,\ \Phi_{20}^*$, and Φ_3^*, respectively. Actually, this is a feasible adaptive control strategy. However, in order to update controller parameters in (16), Assumption (A4b) is needed for developing a stable adaptive law. That is, this control law cannot remove the requirement for the sign information of K_p. In fact, there are some other K_p decomposition based parameterized models devoting to develop practical adaptive control laws under weak assumptions on K_p ([15], [18], [19]). However, it is regret that all these control methods still rely on the prior sign information of K_p. It is noted that the LD^*U decomposition based parametrization model (15) exhibits a form of partial decoupling for the system inputs and outputs. Moreover, such a form allows each input to adjust the system behavior based on individual estimates of the scalar d_i^* and the vector parameter ϕ_i^*. This capability is crucial in dealing with the interactions between multiple input-output pairs efficiently, which is especially valuable in high-dimensional systems where computational burdens should be addressed.

C. Controller Structure

Now, we begin to develop a new version of the control law for the system (1) under the Assumption (A4). First, we define

$$\delta_i^* = d_i^*\phi_i^*,\ i = 1, ..., M.$$

For $u(t) = [u_1(t), ..., u_M(t)]^T$, the control law of this paper is designed as

$$u_i(t) = \frac{1}{1 + \alpha_i(t)d_i(t)}(\phi_i^T(t)\chi_i(t) + \alpha_i(t)\delta_i^T(t)\chi_i(t)) \quad (17)$$

for $i = 1, ..., M$, where $\phi_i(t), \delta_i(t), d_i(t)$ are estimates of $\phi_i^*, \delta_i^*, d_i^*$, respectively, and $\alpha_i(t) \in \mathbb{R},\ i = 1, ..., M$, are gain functions to be designed later to ensure $u_i(t),\ i = 1, ..., M$, are nonsingular in the control process.

Remark 3: From the definitions of $\chi_i(t), i = 1, ..., M$, in (14), we see that the signal $\chi_{i-1}(t)$ contains $u_i(t)$ for $i = 2, ..., M$. Therefore, to ensure a reasonable casuality in control design, it is necessary to initially calculate the control input $u_M(t)$ using accessible $\chi_M(t)$. Once $u_M(t)$ is determined, the signal $\chi_{M-1}(t)$ becomes available. Subsequently, we can calculate $u_{M-1}(t)$ by using $\chi_{M-1}(t)$. Following this recursive process, all control inputs $u_M(t), ..., u_1(t)$ can be obtained. Moreover, to guarantee that the control inputs $u_i(t), i = 1, ..., M$, are nonsingular, it is essential to design $\alpha_i(t)$ in a way such that $1 + \alpha_i(t)d_i(t) \neq 0$ for $i = 1, ..., M$, which will be elaborated subsequently.

D. Tracking Error Equation

To formulate a parameter update law for the estimated parameters within the controller, it is imperative to deduce the tracking error equation using the proposed control law (17). First, by rearranging (17) we get

$$u_i(t) = \phi_i^T(t)\chi_i(t) + \alpha_i(t)\delta_i^T(t)\chi_i(t) - \alpha_i(t)d_i(t)u_i(t), \quad (18)$$

for $i=1,...,M$. For $i=1$, it follows from (15) and (18) that

$$\begin{aligned} u_1(t) &= \phi_1^T(t)\chi_1(t)+\alpha_1(t)\delta_1^T(t)\chi_1(t)-\alpha_1(t)d_1(t)u_1(t) \\ &\quad +\alpha_1(t)e_{\xi 1}(t)-\alpha_1(t)e_{\xi 1}(t) \\ &= \phi_1^T(t)\chi_1(t)-\alpha_1(t)(-\delta_1^T(t)\chi_1(t)+d_1(t)u_1(t)) \\ &\quad +\alpha_1(t)(d_1^*u_1(t)-\delta_1^{*T}\chi_1(t))-\alpha_1(t)e_{\xi 1}(t) \\ &= \phi_1^T(t)\chi_1(t)+\alpha_1(t)\tilde{\delta}_1^T(t)\chi_1(t)-\alpha_1(t)\tilde{d}_1(t)u_1(t) \\ &\quad -\alpha_1(t)e_{\xi 1}(t), \end{aligned} \tag{19}$$

where $\tilde{\delta}_1(t)\triangleq\delta_1(t)-\delta_1^*$ and $\tilde{d}_1(t)\triangleq d_1(t)-d_1^*$. For $i=2,...,M$, still considering (15) and (18), it yields

$$\begin{aligned} u_i(t) &= \phi_i^T(t)\chi_i(t)+\alpha_i(t)\delta_i^T(t)\chi_i(t)-\alpha_i(t)d_i(t)u_i(t) \\ &\quad +\alpha_i(t)(e_{\xi i}(t)+\theta_i^{*T}e_{\theta i}(t)) \\ &\quad -\alpha_i(t)(e_{\xi i}(t)+\theta_i^{*T}e_{\theta i}(t)) \\ &= \phi_i^T(t)\chi_i(t)-\alpha_i(t)(-\delta_i^T(t)\chi_i(t)+d_i(t)u_i(t)) \\ &\quad +\alpha_i(t)(d_i^*u_i(t)-\delta_i^{*T}\chi_i(t)) \\ &\quad -\alpha_i(t)(e_{\xi i}(t)+\theta_i^{*T}e_{\theta i}(t)) \\ &= \phi_i^T(t)\chi_i(t)+\alpha_i(t)\tilde{\delta}_i^T(t)\chi_i(t)-\alpha_i(t)\tilde{d}_i(t)u_i(t) \\ &\quad -\alpha_i(t)(e_{\xi i}(t)+\theta_i^{*T}e_{\theta i}(t)), \end{aligned} \tag{20}$$

where $\tilde{\delta}_i(t)\triangleq\delta_i(t)-\delta_i^*$ and $\tilde{d}_i(t)\triangleq d_i(t)-d_i^*$. Note that $d_i^*\neq 0$ for $i=1,...,M$. Define

$$\rho_i^*=\frac{1}{d_i^*}, i=1,...,M.$$

Then, it follows from (15) that

$$\begin{aligned} &\rho_1^*e_{\xi 1}(t)=u_1(t)-\phi_1^{*T}\chi_1(t), \\ &\rho_i^*(e_{\xi i}(t)+\theta_i^{*T}e_{\theta i}(t))=u_i(t)-\phi_i^{*T}\chi_i(t), i=2,...,M. \end{aligned} \tag{21}$$

Combining (19), (20) and (21), we obtain

$$\begin{aligned} &(\rho_1^*+\alpha_1(t))e_{\xi 1}(t) \\ &=\tilde{\phi}_1^T(t)\chi_1(t)+\alpha_1(t)\tilde{\delta}_1^T(t)\chi_1(t)-\alpha_1(t)\tilde{d}_1(t)u_1(t), \end{aligned}$$

and for $i=2,...,M$,

$$\begin{aligned} &(\rho_i^*+\alpha_i(t))(e_{\xi i}(t)+\theta_i^{*T}e_{\theta i}(t)) \\ &=\tilde{\phi}_i^T(t)\chi_i(t)+\alpha_i(t)\tilde{\delta}_i^T(t)\chi_i(t)-\alpha_i(t)\tilde{d}_i(t)u_i(t). \end{aligned}$$

Denoting $\rho_i(t)$ as an estimate of ρ^* and $\tilde{\rho}_i(t)\triangleq\rho_i(t)-\rho_i^*,\ i=1,...,M$, we have

$$\begin{aligned} &(\rho_1(t)+\alpha_1(t))e_{\xi 1}(t) \\ &=\tilde{\phi}_1^T(t)\chi_1(t)+\alpha_1(t)\tilde{\delta}_1^T(t)\chi_1(t)-\alpha_1(t)\tilde{d}_1(t)u_1(t) \\ &\quad +\tilde{\rho}_1(t)e_{\xi 1}(t), \end{aligned}$$

and

$$\begin{aligned} &(\rho_i(t)+\alpha_i(t))(e_{\xi i}(t)+\theta_i^{*T}e_{\theta i}(t)) \\ &=\tilde{\phi}_i^T(t)\chi_i(t)+\alpha_i(t)\tilde{\delta}_i^T(t)\chi_i(t)-\alpha_i(t)\tilde{d}_i(t)u_i(t) \\ &\quad +\tilde{\rho}_i(t)(e_{\xi i}(t)+\theta_i^{*T}e_{\theta i}(t)),\ i=2,...,M. \end{aligned}$$

Given $\rho_i(t)+\alpha_i(t)\neq 0,\ i=1,...,M$, and defining $\lambda_i^*=\rho_i^*\theta_i^*,\ i=2,...,M$, we get

$$\begin{aligned} e_{\xi 1}(t) &= \tilde{\phi}_1^T(t)\frac{\chi_1(t)}{\rho_1(t)+\alpha_1(t)}+\tilde{\delta}_1^T(t)\frac{\alpha_1(t)\chi_1(t)}{\rho_1(t)+\alpha_1(t)} \\ &\quad -\tilde{d}_1(t)\frac{\alpha_1(t)u_1(t)}{\rho_1(t)+\alpha_1(t)}+\tilde{\rho}_1(t)\frac{e_{\xi 1}(t)}{\rho_1(t)+\alpha_1(t)}, \end{aligned} \tag{22}$$

and for $i=2,...,M$,

$$\begin{aligned} &e_{\xi i}(t)+\theta_i^{*T}\frac{\alpha_i(t)e_{\theta i}(t)}{\rho_i(t)+\alpha_i(t)}+\lambda_i^{*T}\frac{e_{\theta i}(t)}{\rho_i(t)+\alpha_i(t)} \\ &=\tilde{\phi}_i^T(t)\frac{\chi_i(t)}{\rho_i(t)+\alpha_i(t)}+\tilde{\delta}_i^T(t)\frac{\alpha_i(t)\chi_i(t)}{\rho_i(t)+\alpha_i(t)} \\ &\quad -\tilde{d}_i(t)\frac{\alpha_i(t)u_i(t)}{\rho_i(t)+\alpha_i(t)}+\tilde{\rho}_i(t)\frac{e_{\xi i}(t)}{\rho_i(t)+\alpha_i(t)}. \end{aligned} \tag{23}$$

For $i=1,...,M$, define

$$\begin{aligned} \beta_i^* &= [\phi_i^{*T},\delta_i^{*T},d_i^*,\rho_i^*]^T, \\ \beta_i(t) &= [\phi_i^T(t),\delta_i^T(t),d_i(t),\rho_i(t)]^T, \\ \tilde{\beta}_i(t) &= \beta_i(t)-\beta_i^*, \\ \psi_i(t) &= \left[\frac{\chi_i(t)}{\rho_i(t)+\alpha_i(t)},\frac{\alpha_i(t)\chi_i(t)}{\rho_i(t)+\alpha_i(t)},-\frac{\alpha_i(t)u_i(t)}{\rho_i(t)+\alpha_i(t)},\right. \\ &\quad \left.\frac{e_{\xi i}(t)}{\rho_i(t)+\alpha_i(t)}\right]^T, \end{aligned} \tag{24}$$

and for $i=2,...,M$, define

$$\eta_i(t)=\frac{\alpha_i(t)e_{\theta i}(t)}{\rho_i(t)+\alpha_i(t)},\ \zeta_i(t)=\frac{e_{\theta i}(t)}{\rho_i(t)+\alpha_i(t)}. \tag{25}$$

Then, with above definitions, it follows from (22) and (23) that

$$\begin{aligned} &e_{\xi 1}(t)=\tilde{\beta}_1^T(t)\psi_1(t), \\ &e_{\xi i}(t)+\theta_i^{*T}\eta_i(t)+\lambda_i^{*T}\zeta_i(t)=\tilde{\beta}_i^T(t)\psi_i(t), i=2,...,M. \end{aligned} \tag{26}$$

This is the closed-loop tracking error equation for the system (1) and the reference model (2) under the control law (17).

Lemma 1 describes the role of the modified left interactor matrix $\xi_m(z)$ as representing the delay structure of the system transfer matrix $G(z)$. For instance, if $\xi_m(z)=\text{diag}\{z^{d_1},\ldots,z^{d_M}\}$, it follows from (13) that $e_{\xi_i}(t)=e_i(t+d_i)$, which indicates the presence of a time delay d_i in the i-th input-output channel. Thus, considering the input-output delay introduced by the modified left interactor matrix $\xi_m(z)$, we need to further conduct some operations on the tracking error equation (26). For $i=1,...,M$, we select stable polynomial $f_i(z)$ such that its degree is equal to the maximum of the degrees of the polynomials $d_j(z)$ and $h_{kl}^m(z),\ j=1,...,i,\ k=2,...,i,\ l=1,...,k-1$, and it contains $d_i(z)$ as a factor. With chosen $f_i(z)$, we introduce the following filter operators

$$h_i(z)=\frac{1}{f_i(z)},\ i=1,...,M. \tag{27}$$

Denote $H(z)=\text{diag}\{h_1(z),...,h_M(z)\}$. Further, we define the filtered tracking error $\bar{e}(t)$ as

$$\bar{e}(t)=H(z)\xi_m(z)[y-y_m](t)=[\bar{e}_1(t),...,\bar{e}_M(t)]^T. \tag{28}$$

For the sake of simplification, the stable polynomials $f_i(z)$ can be easily selected as $f_i(z)=z^{n_i}$ for a specified degree $n_i,\ i=1,...,M$. A consistent choice of the filter operators is $h_i(z)=1/f(z)$ for $i=1,\ldots,M$. Although this choice is simpler, it may lead to a higher-order filter, given that $f(z)$ is a stable and monic polynomial whose degree is equal to that of the modified left interactor matrix $\xi_m(z)$. Operating both sides of (26) with corresponding filter operator $h_i(z)$ defined in (27), we get

$$\begin{aligned}&\bar{e}_1(t) = h_1(z)[\tilde{\beta}_1^T\psi_1](t),\\&\bar{e}_i(t) + \theta_i^{*T}h_i(z)[\eta_i](t) + \lambda_i^{*T}h_i(z)[\zeta_i](t)\\&= h_i(z)[\tilde{\beta}_i^T\psi_i](t),\ i = 2, ..., M. \end{aligned}\tag{29}$$

Now, we get the desired filtered tracking error equation (29). Notably, based on the definitions of the filter operators $h_i(z),\ i = 1, ..., M$, it can be observed from (28) that the filtered tracking error $\bar{e}(t)$ is available at the current moment. This is crucial for the derivation of a suitable estimation error model and a reasonable parameter update law. In addition, the structures of signals $\psi_i(t),\ i = 1, ..., M,\ \eta_i(t)\ i = 2, ..., M,\ \zeta_i(t),\ i = 2, ..., M$, require $\rho_i(t)+\alpha_i(t) \neq 0,\ i = 1, ..., M$, in the adaptive process. This would be ensured in the subsequent analysis, where a unified singularity-free design of control law (17) and parameter update law to be developed will be conducted.

Remark 4: To handle unknown signs of the high-frequency gain matrix, the key idea is to construct a linear parameter estimation error equation, based on which the parameter update law can be directly designed by using gradient or least-squares techniques. To this end, we propose a new form of control law (17). Leveraging the parametrization equations (15) and (21), we derive the closed-loop tracking error equation (26). Notably, the sign information of the high-frequency gain matrix is implicitly embedded in parameters β_i^* and λ_i^*, which avoids the multiplication of unknown high-frequency gain matrix signs and unknown parameters in the tracking error equation (26). In other words, the tracking error equation (26) is linear with respect to unknown parameters. In the following, we will demonstrate that the estimation error model takes the form of a linear regression and does not need to introduce the sign information of K_p in the parameter update law.

E. Parameter Update Law

Next, we proceed to develop a gradient based parameter update law using an estimation error cost criterion. To do so, we define an estimation error $\epsilon(t) = [\epsilon_1(t), \ldots, \epsilon_M(t)]^T$, which measures the discrepancy between the estimated and actual parameters. Considering the filtered tracking error equation (29), we estimate $\bar{e}_i(t)$ by substituting the true parameters with their estimates. Consequently, the estimation error is expressed as follows:

$$\begin{aligned}&\epsilon_1(t) = \bar{e}_1(t) - h_1(z)[\beta_1^T\psi_1](t) + \beta_1^T(t)h_1(z)[\psi_1](t),\\&\epsilon_i(t) = \bar{e}_i(t) + \theta_i^T(t)h_i(z)[\eta_i](t) + \lambda_i^T(t)h_i(z)[\zeta_i](t)\\&\quad -h_i(z)[\beta_i^T\psi_i](t)+\beta_i^T(t)h_i(z)[\psi_i](t),\ i=2, ..., M,\end{aligned}\tag{30}$$

where $\theta_i(t), \lambda_i(t)$ denote estimates of $\theta_i^*, \lambda_i^*,\ i = 2, ..., M$, respectively. This together with the filtered tracking error equation (29) yields the estimation error equation as

$$\begin{aligned}&\epsilon_1(t) = \tilde{\beta}_1^T(t)\bar{\psi}_1(t),\\&\epsilon_i(t) = \tilde{\theta}_i^T(t)\bar{\eta}_i(t) + \tilde{\lambda}_i^T(t)\bar{\zeta}_i(t) + \tilde{\beta}_i^T(t)\bar{\psi}_i(t),\ i=2, ..., M,\end{aligned}\tag{31}$$

where $\tilde{\theta}_i(t) \triangleq \theta_i(t)-\theta_i^*,\ i = 2, ..., M,\ \tilde{\lambda}_i(t) \triangleq \lambda_i(t)-\lambda_i^*,\ i = 2, ..., M$, and

$$\begin{aligned}\bar{\psi}_i(t) &\triangleq h_i(z)[\psi_i](t),\ i = 1, ..., M,\\\bar{\eta}_i(t) &\triangleq h_i(z)[\eta_i](t),\ i = 2, ..., M,\\\bar{\zeta}_i(t) &\triangleq h_i(z)[\zeta_i](t),\ i = 2, ..., M.\end{aligned}\tag{32}$$

Based on the estimation error equation (31), we introduce the following quadratic cost function

$$J = \frac{1}{2m^2}\sum_{i=1}^{M}\epsilon_i^2,$$

where $m = m(t)$ is a normalized signal to be defined. Then, we derive the gradient of J with respect to $\beta_i(t)$, $\theta_i(t)$ and $\lambda_i(t)$ as

$$\begin{aligned}\frac{\partial J}{\partial \beta_i} &= \frac{\epsilon_i(t)\bar{\psi}_i(t)}{m^2(t)},\ i = 1, ..., M,\\\frac{\partial J}{\partial \theta_i} &= \frac{\epsilon_i(t)\bar{\eta}_i(t)}{m^2(t)},\ i = 2, ..., M,\\\frac{\partial J}{\partial \lambda_i} &= \frac{\epsilon_i(t)\bar{\zeta}_i(t)}{m^2(t)},\ i = 2, ..., M,\end{aligned}$$

which motivates a gradient based parameter update law as

$$\begin{aligned}\beta_i(t+1) &= \beta_i(t) - \frac{\Gamma_{\beta i}\epsilon_i(t)\bar{\psi}_i(t)}{m^2(t)},\ i = 1, ..., M,\\\theta_i(t+1) &= \theta_i(t) - \frac{\Gamma_{\theta i}\epsilon_i(t)\bar{\eta}_i(t)}{m^2(t)},\ i = 2, ..., M,\\\lambda_i(t+1) &= \lambda_i(t) - \frac{\Gamma_{\lambda i}\epsilon_i(t)\bar{\zeta}_i(t)}{m^2(t)},\ i = 2, ..., M\end{aligned}\tag{33}$$

with

$$m = \sqrt{1 + \sum_{i=1}^{M}\bar{\psi}_i^T\bar{\psi}_i + \sum_{i=2}^{M}\bar{\eta}_i^T\bar{\eta}_i + \sum_{i=2}^{M}\bar{\zeta}_i^T\bar{\zeta}_i}\tag{34}$$

and $\Gamma_{\beta i}, \Gamma_{\theta i}, \Gamma_{\lambda i}$ being adaptive gains such that $0 < \Gamma_{\beta i} = \Gamma_{\beta i}^T < 2I_{4vM+2M+2-2i}$, $0 < \Gamma_{\theta i} = \Gamma_{\theta i}^T, \Gamma_{\lambda i} = \Gamma_{\lambda i}^T < 2I_{i-1}$.

Remark 5: It is noteworthy that we define the estimation error $\epsilon(t)$ by use of the filtered tracking error $\bar{e}(t)$ and filtered regressor signals, all of which are available at the current time instant. Therefore, the estimation error $\epsilon(t)$ is utilizable in designing the parameter update law (33). This indicates that the reasonable causality of adaptive control design is guaranteed by introducing filtered signals. Specifically, the modified left interactor matrix $\xi_m(z)$ makes some signals unavailable, such as $e_\xi(t), e_{\theta_i}(t)$, because they include future time output signals. It is necessary to introduce the filter operators $h_i(z),\ i = 1, ..., M$, to deal with these unmeasurable signals. Failure to do so may lead to casuality contradiction problem in control design, which is a challenge often encountered in discrete-time adaptive control systems.

Remark 6: The definitions of $d_i^*,\ i = 1, ..., M$, indicate that their sign information can be derived by the signs of leading principal minors of K_p. These signs, along with the upper bound knowledge $d_i^0,\ i = 1, ..., M$, are assumed to be known in (A4b). In some traditional adaptive control schemes, this information is crucial: the sign of d_i^* determines the direction of parameter updates, and d_i^0 is used to select the adaptive gain to ensure the stability of the update law. In contrast, our method does not need such requirements on K_p in the parameter update law (33). Therefore, the proposed control law (17) and parameter update law (33) completely remove the need for knowledge of $\text{sign}[d_i^*]$ and d_i^0 as stated in (A4b). This is because the estimation error model (31) takes a linear regression form with respect to the unknown parameters, differing from the bilinear form typically seen in traditional estimation error models (see [15], [16] for further details).

Remark 7: The proposed approach removes the need for prior knowledge of the signs of K_p by introducing an extra parameter estimate $\delta_i(t)$ in the controller (17). Thus, the dimensions of parameter vectors to be estimated are slightly higher than some traditional MRAC design schemes. Additionally, since no excitation condition is imposed on the regressor signals, the increase of parameter dimension does not influence transient adaptation.

The following lemma elucidates that the parameter update law (33) possesses certain desirable properties with respect to the estimated parameters.

Lemma 3: *The parameter update law (33) ensures that*

(i) $\beta_i(t) \in L^\infty,\ i=1,...,M,\ \theta_i(t) \in L^\infty,\ i=2,...,M,\ \lambda_i(t) \in L^\infty,\ i=2,...,M$;

(ii) $\frac{\epsilon_i(t)}{m(t)} \in L^2 \cap L^\infty,\ i=1,...,M$; *and*

(iii) $\beta_i(t+i_0)-\beta_i(t) \in L^2\cap L^\infty,\ i=1,...,M,\ \theta_i(t+i_0)-\theta_i(t) \in L^2\cap L^\infty,\ i=2,...,M,\ \lambda_i(t+i_0)-\lambda_i(t) \in L^2\cap L^\infty,\ i=2,...,M$, *for any integer* $i_0>0$.

Proof. Consider the following positive definite function

$$V(\tilde{\beta}_i,\tilde{\theta}_i,\tilde{\lambda}_i) = \sum_{i=1}^{M}\tilde{\beta}_i^T\Gamma_{\beta i}^{-1}\tilde{\beta}_i + \sum_{i=2}^{M}\tilde{\theta}_i^T\Gamma_{\theta i}^{-1}\tilde{\theta}_i + \sum_{i=2}^{M}\tilde{\lambda}_i^T\Gamma_{\lambda i}^{-1}\tilde{\lambda}_i.$$

Then, it follows from (33) that the time increment of V is

$$\begin{aligned} & V(\tilde{\beta}_i(t+1),\tilde{\theta}_i(t+1),\tilde{\lambda}_i(t+1)) - V(\tilde{\beta}_i(t),\tilde{\theta}_i(t),\tilde{\lambda}_i(t)) \\ =& -\sum_{i=1}^{M}\left(2-\frac{\bar{\psi}_i^T(t)\Gamma_{\beta i}\bar{\psi}_i(t)}{m^2(t)}\right)\frac{\epsilon_i^2(t)}{m^2(t)} \\ & -\sum_{i=2}^{M}\left(2-\frac{\bar{\eta}_i^T(t)\Gamma_{\theta i}\bar{\eta}_i(t)}{m^2(t)}\right)\frac{\epsilon_i^2(t)}{m^2(t)} \\ & -\sum_{i=2}^{M}\left(2-\frac{\bar{\zeta}_i^T(t)\Gamma_{\lambda i}\bar{\zeta}_i(t)}{m^2(t)}\right)\frac{\epsilon_i^2(t)}{m^2(t)}. \end{aligned}$$

From the conditions of $\Gamma_{\beta i},\Gamma_{\theta i},\Gamma_{\lambda i}$ and the definition of $m(t)$ in (34), we obtain $2-\frac{\bar{\psi}_1^T\Gamma_{\beta 1}\bar{\psi}_1}{m^2}>0$ and $2-\left(\frac{\bar{\psi}_i^T\Gamma_{\beta i}\bar{\psi}_i}{m^2}+\frac{\bar{\eta}_i^T\Gamma_{\theta i}\bar{\eta}_i}{m^2}+\frac{\bar{\zeta}_i^T\Gamma_{\lambda i}\bar{\zeta}_i}{m^2}\right)>0,\ i=2,...,M$. Then, for some constants $k_i,\ i=1,...,M$, such that $0<k_1<2-\frac{\bar{\psi}_1^T\Gamma_{\beta 1}\bar{\psi}_1}{m^2}$ and $0<k_i<2-\left(\frac{\bar{\psi}_i^T\Gamma_{\beta i}\bar{\psi}_i}{m^2}+\frac{\bar{\eta}_i^T\Gamma_{\theta i}\bar{\eta}_i}{m^2}+\frac{\bar{\zeta}_i^T\Gamma_{\lambda i}\bar{\zeta}_i}{m^2}\right),\ i=2,...,M$, we obtain $V(\tilde{\beta}_i(t+1),\tilde{\theta}_i(t+1),\tilde{\lambda}_i(t+1))-V(\tilde{\beta}_i(t),\tilde{\theta}_i(t),\tilde{\lambda}_i(t)) \le -\sum_{i=1}^{M}\frac{k_i\epsilon_i^2(t)}{m^2(t)}$. This implies that $\beta_i(t),\ \theta_i(t),\ \lambda_i(t)\in L^\infty$ and $\frac{\epsilon_i(t)}{m(t)}\in L^2$. From (31), we have $\frac{\epsilon_i(t)}{m(t)}\in L^\infty,\ i=1,...,M$. From (33), we have $\beta_i(t+1)-\beta_i(t),\ \theta_i(t+1)-\theta_i(t),\ \lambda_i(t+1)-\lambda_i(t)\in L^2\cap L^\infty$. Finally, using the inequalities

$$\begin{aligned} ||\beta_i(t+i_0)-\beta_i(t)||_2 &\le \sum_{k=0}^{i_0-1}||\beta_i(t+k+1)-\beta_i(t+k)||_2, \\ ||\theta_i(t+i_0)-\theta_i(t)||_2 &\le \sum_{k=0}^{i_0-1}||\theta_i(t+k+1)-\theta_i(t+k)||_2, \\ ||\lambda_i(t+i_0)-\lambda_i(t)||_2 &\le \sum_{k=0}^{i_0-1}||\lambda_i(t+k+1)-\lambda_i(t+k)||_2, \end{aligned}$$

we obtain $\beta_i(t+i_0)-\beta_i(t),\theta_i(t+i_0)-\theta_i(t),\lambda_i(t+i_0)-\lambda_i(t)\in L^2\cap L^\infty$ for any integer $i_0>0$. This completes the proof. □

This lemma demonstrates that the parameter update law (33) ensures certain desired properties for the parameter estimates. These properties are useful for the subsequent analysis of the closed-loop system stability. Notably, although Lemma 3 implies that $\lim_{t\to\infty}(\beta_i(t+1)-\beta_i(t))=0$ for $i=1,\dots,M$, $\lim_{t\to\infty}(\theta_i(t+1)-\theta_i(t))=0$ for $i=2,\dots,M$, and $\lim_{t\to\infty}(\lambda_i(t+1)-\lambda_i(t))=0$ for $i=2,\dots,M$, this does not necessarily guarantee that the parameter estimates $\beta_i(t),\theta_i(t)$, and $\lambda_i(t)$ converge to their true values $\beta_i^*,\theta_i^*,\lambda_i^*$, respectively. This is because the proposed control technique does not rely on any form of excitation condition. Nevertheless, the asymptotic output tracking objective can be achieved, which will be proved in the following main result.

F. Singularity-Free Design

From the structures of the control law (17) and the parameter update law (33), it is evident that the control algorithm may blow up if $1+\alpha_i(t)d_i(t)=0$ or $\alpha_i(t)+\rho_i(t)=0$ occurs during the control process. Therefore, to ensure that both the adaptive control law and the parameter update law remain nonsingular at all times, we need to design $\alpha_i(t)$ to guarantee

$$1+\alpha_i(t)d_i(t)\neq 0,\ i=1,...,M, \tag{35}$$

$$\alpha_i(t)+\rho_i(t)\neq 0,\ i=1,...,M, \tag{36}$$

for any possible real value $d_i(t)$ and $\rho_i(t)$. To achieve this, the gain functions $\alpha_i(t),\ i=1,...,M$, are designed as

$$\alpha_i(t)=\begin{cases} -(|\rho_i(t)|+\underline{\alpha}_i), & d_i(t)<0, \\ |\rho_i(t)|+\underline{\alpha}_i, & d_i(t)\ge 0, \end{cases} \tag{37}$$

where $\underline{\alpha}_i,\ i=1,...,M$, are arbitrary positive constants to be chosen. The following lemma shows that no singularity problem would arise in the adaptive process with $\alpha_i(t),\ i=1,...,M$, defined as (37).

Lemma 4: *The gain functions $\alpha_i(t),\ i=1,...,M$, designed in (37) ensure that (35) and (36) always hold.*

Proof. Firstly, we prove the condition (36). When $d_i(t)<0$, with the definition of $\alpha_i(t)$ in (37), we obtain

$$\begin{aligned} &\alpha_i(t)+\rho_i(t) = -|\rho_i(t)|-\underline{\alpha}_i+\rho_i(t) \\ =&\begin{cases} -\underline{\alpha}_i, & \rho_i(t)\ge 0, \\ 2\rho_i(t)-\underline{\alpha}_i, & \rho_i(t)<0, \end{cases}\quad i=1,...,M. \end{aligned}$$

Due to $\underline{\alpha}_i > 0$, it yields $\alpha_i(t) + \rho_i(t) < 0$ when $d_i(t) < 0$ for $i = 1, ..., M$. Similarly, when $d_i(t) \geq 0$, we get

$$\alpha_i(t) + \rho_i(t) = |\rho_i(t)| + \underline{\alpha}_i + \rho_i(t)$$
$$= \begin{cases} 2\rho_i(t) + \underline{\alpha}_i, & \rho_i(t) \geq 0, \\ \underline{\alpha}_i, & \rho_i(t) < 0, \end{cases} \quad i = 1, ..., M.$$

This implies that $\alpha_i(t)+\rho_i(t) > 0$ with arbitrary $\underline{\alpha}_i > 0$ when $d_i(t) \geq 0$ for $i = 1, ..., M$. Hence, (36) holds.

Then, we consider the condition (35). When $d_i(t) < 0$, it follows from (37) that

$$1 + \alpha_i(t)d_i(t) = 1 - (|\rho_i(t)| + \underline{\alpha}_i)\, d_i(t)$$
$$= 1 - |\rho_i(t)|d_i(t) - \underline{\alpha}_i d_i(t), \ i = 1, ..., M.$$

Since $d_i(t) < 0$, we get $1-|\rho_i(t)|d_i(t)-\underline{\alpha}_i d_i(t) > 0$ is always true with arbitrary $\underline{\alpha}_i > 0$ for $i = 1, ..., M$. When $d_i(t) \geq 0$, we have

$$1 + \alpha_i(t)d_i(t) = 1 + (|\rho_i(t)| + \underline{\alpha}_i)\, d_i(t)$$
$$= 1 + |\rho_i(t)|d_i(t) + \underline{\alpha}_i d_i(t), \ i = 1, ..., M.$$

It follows from $\underline{\alpha}_i > 0$ that $1 + |\rho_i(t)|d_i(t) + \underline{\alpha}_i d_i(t) > 0$ is always true when $d_i(t) \geq 0$ for $i = 1, ..., M$. Thus, (35) holds. This completes the proof. □

Remark 8: Different from the traditional control laws (8) and (16), this paper introduces a new form of the controller (17). The derived gradient based parameter update law (33) no longer necessitates prior sign information of K_p. However, the inclusion of denominator terms $1+\rho_i(t)d_i(t)$ and $\alpha_i(t)+\rho_i(t)$ may cause singularity problem in the adaptive process. To address this concern, we introduce time-varying gain functions $\alpha_i(t), \ i = 1, ..., M,$ as defined in (37), achieving a unified singularity-free design for the control law (17) and the parameter update law (33). As demonstrated in Lemma 4, the well-designed $\alpha_i(t), i = 1, ..., M,$ ensure that $1 + \rho_i(t)d_i(t)$ and $\alpha_i(t) + \rho_i(t)$ are always non-zero. Therefore, the control law (17) and the parameter update law (33) remain implementable throughout the control process.

G. Stability Analysis

Based on above derivations, we are ready to present the main result of this paper as follows.

Theorem 1: *Under Assumptions (A1)-(A4), the adaptive control law (17) along with the parameter update law (33), applied to the system (1), ensures the closed-loop system is stable and the system output $y(t)$ tracks the reference output $y^*(t)$ asymptotically, i.e., $\lim_{t\to\infty}(y(t) - y^*(t)) = 0$.*

Proof. First, we prove some bounded properties of the regressor vectors $\bar{\psi}_i(t), \ i = 1, ..., M, \ \bar{\eta}_i(t), \ i = 2, ..., M,$ and $\bar{\zeta}_i(t), \ i = 2, ..., M$. From the definition of $\psi_i(t)$ in (24) and the bound property of the estimated parameters demonstrated in Lemma 3, we have $||\psi_i(t)|| \leq c \max_{k\leq t} ||u(k)|| + c \max_{k\leq t} ||y(k)||+c \max_{k\leq t} ||e_\xi(k)||+c, \ i = 1, ..., M,$ where $c > 0$ denotes a generic signal bound. Since $y^*(t)$ is bounded, we get

$$||\psi_i(t)|| \leq c \max_{k\leq t} ||u(k)|| + c \max_{k\leq t} ||e(k)|| + c \max_{k\leq t} ||e_\xi(k)|| + c, \ i = 1, ..., M. \quad (38)$$

Since $e(t) = \xi_m^{-1}(z)[e_\xi](t)$ and $\xi_m^{-1}(z)$ is proper and stable, we have

$$||e(t)|| \leq c \max_{k\leq t} ||e_\xi(k)|| + c. \quad (39)$$

Then, for $i = 1, ..., M$, it follows from (38) and (39) that

$$||\psi_i(t)|| \leq c \max_{k\leq t} ||u(k)|| + c \max_{k\leq t} ||e_\xi(k)|| + c. \quad (40)$$

From the system model (1), we have $u(t) = G^{-1}(z)[y](t) = (\xi_m(z)G(z))^{-1}\xi_m(z)[y](t)$. From Lemma 1, we obtain that $(\xi_m(z)G(z))^{-1}$ is proper. From Assumption (A1) and the definition of $\xi_m(z)$, we get $(\xi_m(z)G(z))^{-1}$ is stable. Then, combing the bounded property of $y^*(t)$, we have

$$||u(t)|| \leq c \max_{k\leq t} ||\xi_m(z)[y](k)|| + c \leq c \max_{k\leq t} ||e_\xi(k)|| + c. \quad (41)$$

Combining (40) and (41), we get

$$||\psi_i(t)|| \leq c \max_{k\leq t} ||e_\xi(k)|| + c, \ i = 1, ..., M. \quad (42)$$

Since $\bar{\psi}_i(t) = h_i(z)[\psi_i](t)$ and $h_i(z) = \frac{1}{f_i(z)}$ with $f_i(z)$ being stable, we obtain $||\bar{\psi}_i(t)|| \leq c \max_{k\leq t} ||\psi_i(k)||+c, \ i = 1, ..., M$. This together with (42) gives

$$||\bar{\psi}_i(t)|| \leq c \max_{k\leq t} ||e_\xi(k)|| + c, \ i = 1, ..., M. \quad (43)$$

Similarly, leveraging the definitions of $\bar{\eta}_i(t)$ and $\bar{\zeta}_i(t)$ along with the bounded properties of the estimated parameters provided in Lemma 3, we derive

$$||\bar{\eta}_i(t)|| \leq c \max_{k\leq t} ||e_\xi(k)|| + c, \ i = 2, ..., M,$$
$$||\bar{\zeta}_i(t)|| \leq c \max_{k\leq t} ||e_\xi(k)|| + c, \ i = 2, ..., M. \quad (44)$$

Next, we establish the stability of the closed-loop system. Without loss of generality, let $f_i(z)$ be selected as z^{n_i} with a specified degree n_i for $i = 1, ..., M$. Utilizing the definition of the estimation error in (30), we obtain

$$\bar{e}_1(t) = \epsilon_1(t) + h_1(z)[\beta_1^T\psi_1] - \beta_1^T(t)h_1(z)[\psi_1](t) = \frac{\epsilon_1(t)}{m(t)}m(t) - (\beta_1(t) - \beta_1(t - n_1))^T\bar{\psi}_1(t), \quad (45)$$

and for $i = 2, ..., M$,

$$\bar{e}_i(t) + \theta_i^T(t)h_i(z)[\eta_i](t) + \lambda_i^T(t)h_i(z)[\zeta_i](t) = \epsilon_i(t) + h_i(z)[\beta_i^T\psi_i] - \beta_i^T(t)h_i(z)[\psi_i](t) = \frac{\epsilon_i(t)}{m(t)}m(t) - (\beta_i(t) - \beta_i(t - n_i))^T\bar{\psi}_i(t). \quad (46)$$

Recalling the definition of $m(t)$ in (34), it follows from (43) and (44) that

$$||m(t)|| \leq 1 + \sum_{i=1}^{M} ||\bar{\psi}_i(t)|| + \sum_{i=2}^{M} ||\bar{\eta}_i(t)|| + \sum_{i=2}^{M} ||\bar{\zeta}_i(t)|| \leq c \max_{k\leq t} ||e_\xi(k)|| + c. \quad (47)$$

Combining (43), (45) and (47), we get

$$\begin{aligned}||\bar{e}_1(t)|| \leq & \left|\left|\frac{\epsilon_1(t)}{m(t)}\right|\right| ||m(t)|| + ||\beta_1(t)-\beta_1(t-n_1)|| ||\bar{\psi}_1(t)|| \\ \leq & c||\beta_1(t)-\beta_1(t-n_1)|| \max_{k\leq t} ||e_\xi(k)|| + c \\ & + c\left|\left|\frac{\epsilon_1(t)}{m(t)}\right|\right| \max_{k\leq t} ||e_\xi(k)||.\end{aligned}$$

According to Lemma 3, we have $\beta_1(t)-\beta_1(t-n_1) \in L^2 \cap L^\infty$ and $\frac{\epsilon_1(t)}{m(t)} \in L^2 \cap L^\infty$. Then, we get

$$||\bar{e}_1(t)|| \leq \tau(t) \max_{k\leq t} ||e_\xi(k)|| + c, \tag{48}$$

where $\tau(t)$ denotes a generic $L^2 \cap L^\infty$ function which converges to zero as t goes to ∞. Recalling the definitions of $\bar{\eta}_2(t)$ and $\bar{\zeta}_2(t)$ in (32), as well as the definitions of $\eta_2(t)$ and $\zeta_2(t)$ in (25), and combining (48), we obtain

$$\begin{aligned}||\bar{\eta}_2(t)|| \leq c||\bar{e}_1(t)|| + c \leq \tau(t) \max_{k\leq t} ||e_\xi(k)|| + c, \\ ||\bar{\zeta}_2(t)|| \leq c||\bar{e}_1(t)|| + c \leq \tau(t) \max_{k\leq t} ||e_\xi(k)|| + c.\end{aligned} \tag{49}$$

Then, it follows from (43), (46), (47), (49) and Lemma 3 that

$$\begin{aligned}& ||\bar{e}_2(t)|| \\ \leq & \left|\left|\frac{\epsilon_2(t)}{m(t)}\right|\right| ||m(t)|| + ||\beta_2(t)-\beta_2(t-n_2)|| ||\bar{\psi}_2(t)|| \\ & + ||\theta_2(t)|| ||\bar{\eta}_2(t)|| + ||\lambda_2(t)|| ||\bar{\zeta}_2(t)|| \\ \leq & \tau(t) \max_{k\leq t} ||e_\xi(k)|| + c.\end{aligned}$$

Similarly, we obtain $||\bar{e}_3(t)|| \leq \tau(t)\max_{k\leq t} ||e_\xi(k)|| + c$. Following this procedure, we recursively derive $||\bar{e}_i(t)|| \leq \tau(t)\max_{k\leq t} ||e_\xi(k)|| + c,\ i = 1, ..., M$, and hence, $||\bar{e}(t)|| \leq \tau(t)\max_{k\leq t} ||e_\xi(k)|| + c$, which implies that $\bar{e}(t)$ is bounded. Therefore, we get $e(t)$ is bounded and in turn $y(t)$ is bounded. Then, it follows from the minimum phase nature of the system (1) that $u(t)$ is also bounded. Furthermore, it can be demonstrated that all closed-loop signals are bounded, which means the closed-loop system is stable.

Finally, we prove the tracking performance. From (45) in which $\frac{\epsilon_1(t)}{m(t)} \in L^2$ and $\beta_1(t)-\beta_1(t-n_1) \in L^2$, we have $\bar{e}_1(t) \in L^2$, that is, $\lim_{t\to\infty} \bar{e}_1(t) = 0$. Considering the definitions of $\bar{\eta}_2(t)$ and $\bar{\zeta}_2(t)$ in (32), it follows from $\lim_{t\to\infty} \bar{e}_1(t) = 0$ that $\bar{\eta}_2(t) \in L^2$ and $\bar{\zeta}_2(t) \in L^2$. Subsequently, utilizing Lemma 3 and (46), we conclude that $\bar{e}_2(t) \in L^2$, that is, $\lim_{t\to\infty} \bar{e}_2(t) = 0$. Following this procedure, we can recursively derive that $\bar{e}_i(t) \in L^2$ and $\lim_{t\to\infty} \bar{e}_i(t) = 0$ for $i = 1, ..., M$. Thus, $\bar{e}(t) \in L^2$ and $\lim_{t\to\infty} \bar{e}(t) = 0$. Due to the stability of the inverse of $\xi_m(z)$, it can be inferred that $\lim_{t\to\infty} e(t) = 0$. This completes the proof. □

So far, we have developed a new adaptive controller (17) along with a parameter updater law (33) for the MIMO discrete-time LTI system (1). The proposed method relies solely on a mild design condition on the high-frequency matrix, and still achieves the same asymptotic tracking performance as some well-developed traditional design schemes. Importantly, the proposed method does not involve multi-model parameter estimation burden and the persistent switching issue.

IV. Simulation study

This section gives two simulation examples, covering a numerical example and an aircraft dynamics model, to showcase the design process and substantiate the theoretical findings.

A. Simulation for A Numerical Example

First, we introduce a numerical example to verify the effectiveness of the proposed control method.

Simulation model. Consider the following system model

$$y(t) = G(z)[u](t) \tag{50}$$

with

$$G(z) = Z(z)P^{-1}(z) = \begin{bmatrix} \frac{1}{z+0.1} & \frac{z-0.3}{(z+0.1)(z+1.1)} \\ 0 & \frac{z+0.8}{(z+0.1)(z+1.1)} \end{bmatrix}, \tag{51}$$

where

$$\begin{aligned}P(z) &= \begin{bmatrix} z+0.1 & 0 \\ 0 & z^2+1.2z+0.11 \end{bmatrix}, \\ Z(z) &= \begin{bmatrix} 1 & z-0.3 \\ 0 & z+0.8 \end{bmatrix}.\end{aligned} \tag{52}$$

Hence, we determine that the modified left interactor matrix of the transfer function $G(z)$ is

$$\xi_m(z) = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}, \tag{53}$$

which has a proper and stable inverse. Subsequently, we calculate the high-frequency gain matrix K_p of $G(z)$ is

$$K_p = \lim_{z\to\infty} \xi_m(z)G(z) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \tag{54}$$

which is nonsingular. From (52), we calculate $\det[Z(z)] = z + 0.8$, which means that the system (50) is minimum phase. Moreover, the observability index of the system (50) is 2. From (54), we calculate the leading principle minors of K_p are $\Delta_1 = 1$ and $\Delta_2 = 1$, which are non-zero. In the simulation example, it is assumed that the constant parameters in $G(z)$ are all unknown. The reference model is selected as $y^*(t) = \xi_m^{-1}(z)[r](t)$, where $r(t) = [4\sin 0.3t, 5\cos 0.3t]^T$ and $\xi_m(z)$ is determined in (53).

Parameter setting. Utilizing the definition in (3) and the observability index of the system (50), we determine $A(z) = I_2$. Choose $\Lambda(z) = z$. Consequently, we derive $\omega_1(t)$ and $\omega_2(t)$ using (3). Subsequently, by employing the matching equation (4), we determine

$$\begin{aligned}\Theta_1^{*T} &= \begin{bmatrix} 0.1 & -3.3 \\ 0 & -0.8 \end{bmatrix}, \Theta_2^{*T} = \begin{bmatrix} -0.01 & 0.45 \\ 0 & -0.8 \end{bmatrix}, \\ \Theta_{20}^* &= \begin{bmatrix} 0 & 4.4 \\ 0 & 1.2 \end{bmatrix}, \Theta_3^* = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.\end{aligned}$$

From (54), we decompose K_p as $K_p = LD^*U$, where

$$L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, D^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then, we obtain

$$\Phi_1^{*T} = \begin{bmatrix} 0.1 & -4.1 \\ 0 & -0.8 \end{bmatrix}, \Phi_2^{*T} = \begin{bmatrix} -0.01 & 0.56 \\ 0 & 0.11 \end{bmatrix},$$
$$\Phi_{20}^* = \begin{bmatrix} 0 & 5.6 \\ 0 & 1.2 \end{bmatrix}, \Phi_3^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Phi_0^* = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and $\Theta_0^* = 0_2, \theta_2^* = 0$. Thus, it yields

$$\phi_1^* = [1, 0.1, -4.1, -0.01, 0.56, 0, 5.6, 1, 0]^T,$$
$$\phi_2^* = [0, -0.8, 0, 0.11, 0, 1.2, 0, 1]^T.$$

With $d_1^* = \Delta_1 = 1$ and $d_2^* = \frac{\Delta_2}{\Delta_1} = 1$, we get $\delta_1^* = \phi_1^*$ and $\delta_2^* = \phi_2^*$. Since $\rho_i^* = \frac{1}{d_i^*}$, we have $\rho_1^* = 1, \rho_2^* = 1$. From $\lambda_2^* = \rho_2^*\theta_2^*$, we have $\lambda_2^* = 0$. From the form of $\omega(t)$, we determine $\chi_2(t) = \omega(t)$. Denote $\phi_2(t), \delta_2(t), d_2(t), \rho_2(t), \theta_2(t), \lambda_2(t)$ as estimates of $\phi_2^*, \delta_2^*, d_2^*, \rho_2^*, \theta_2^*, \lambda_2^*$, respectively, with $\phi_2(t) = [\phi_{21}(t), ..., \phi_{28}(t)]^T$ and $\delta_2(t) = [\delta_{21}(t), ..., \delta_{28}(t)]^T$. Then, according to (17), we calculate $u_2(t)$ with $\alpha_2(t)$ being determined by (37) and $\underline{\alpha}_2 = 0.5$. With $u_2(t)$ to hand, we obtain $\chi_1(t) = [u_2(t), \omega^T(t)]^T$. Denote $\phi_1(t), \delta_1(t), d_1(t), \rho_1(t)$ as estimates of $\phi_1^*, \delta_1^*, d_1^*, \rho_1^*$, respectively, with $\phi_1(t) = [\phi_{11}(t), ..., \phi_{19}(t)]^T$ and $\delta_1(t) = [\delta_{11}(t), ..., \delta_{19}(t)]^T$. Then, we calculate $u_1(t)$ with $\alpha_1(t)$ being determined by (37) and $\underline{\alpha}_1 = 0.5$. Choose

$$\phi_1(0) = [0.5, 0.15, -3, -0.1, 0.4, 0.5, 5, 1.5, 0.5]^T,$$
$$\phi_2(0) = [-0.2, -0.6, 0.1, 0.1, 0.2, 1, 0, 1]^T,$$
$$\delta_1(0) = [0.6, 0.2, -3.8, 0, 0.5, 0.1, 5, 1.2, 0.2]^T,$$
$$\delta_2(0) = [0.3, -1, 0, 0.2, 0, 1, 0, 0.8]^T,$$

and $d_1(0) = 1.3$, $d_2(0) = 0.7$, $\rho_1(0) = 1.6$, $\rho_2(0) = 0.5$, $\lambda_2(0) = 0.2$, $\theta_2(0) = 0.9$. Moreover, we choose adaptive gains $\Gamma_{\beta 1} = 0.9I_{20}, \Gamma_{\beta 2} = 0.9I_{18}, \Gamma_{\theta 2} = 0.9, \Gamma_{\lambda 2} = 0.9$, filter operators $h_1(z) = 1/z, h_2(z) = 1/z$, and $y(0) = [5, -4]^T$. Consequently, the parameter update law can be obtained in (32) with all chosen parameters and regressors.

Simulation figures. The system response is depicted in Figs. 1-5. Fig. 1 presents the system output $y(t)$ versus the reference output $y^*(t)$, which illustrates that $y(t)$ tracks $y^*(t)$ asymptotically. Fig. 2 displays the trajectory of the control input $u(t)$. Fig. 3 shows the trajectories of time-varying gain functions $\alpha_1(t)$ and $\alpha_2(t)$, as well as corresponding singularity-free indices $\alpha_i(t) + \rho_i(t)$ and $1 + \alpha_i(t)d_i(t)$, $i = 1, 2$. From this figure, we see that conditions $\alpha_i(t) + \rho_i(t) \neq 0$ and $1 + \alpha_i(t)d_i(t) \neq 0$, $i = 1, 2$, always hold, which is consistent with the conclusion of Lemma 4. Fig. 4 and Fig. 5 show the trajectories of part of the estimated parameters. All these figures demonstrate that closed-loop signals are bounded. Notably, the proposed control scheme does not depend on any type of excitation condition. Consequently, parameter estimations may not converge to the corresponding true values, as illustrated in Fig. 4 and Fig. 5. Nevertheless, the asymptotic output tracking control objective is still achieved. In conclusion, the simulation results affirm the effectiveness of the proposed output feedback adaptive control strategy.

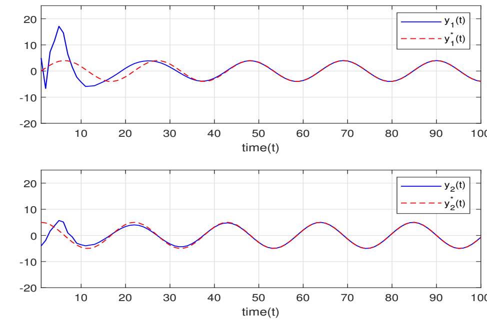

Fig. 1. Trajectories of the output $y(t)$ and the reference output $y^*(t)$

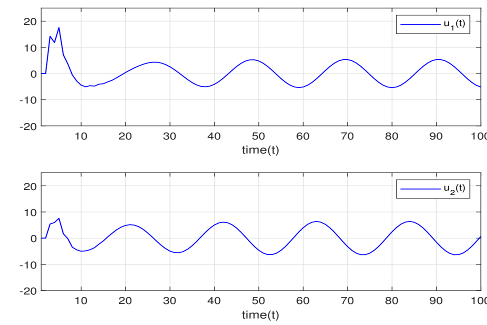

Fig. 2. Trajectory of the control input $u(t)$

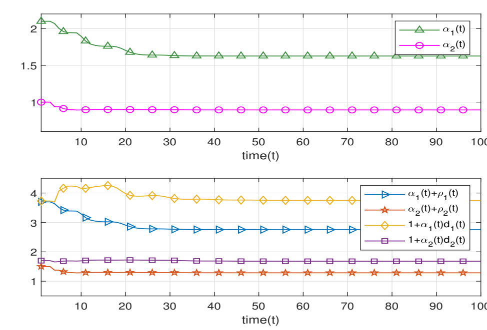

Fig. 3. Trajectories of $\alpha_i(t)$, $\alpha_i(t) + \rho_i(t)$ and $1 + \alpha_i(t)d_i(t)$

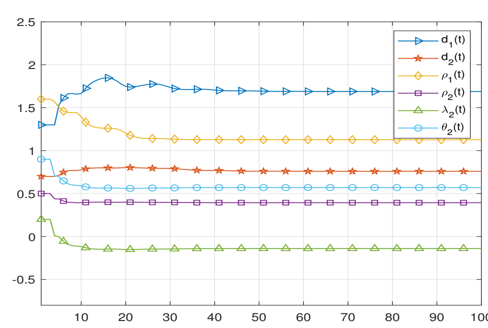

Fig. 4. Trajectories of $d_1(t), d_2(t), \rho_1(t), \rho_2(t), \lambda_2(t), \theta_2(t)$

B. Simulation for A Linearized Aircraft Model

In this subsection, we simulate the linearized lateral dynamics model of the Boeing 747 airplane to verify the validity of

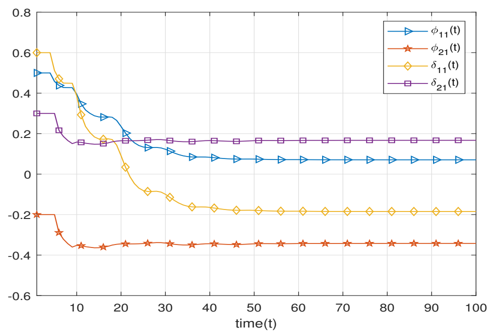

Fig. 5. Trajectories of $\phi_{11}(t), \phi_{21}(t), \delta_{11}(t), \delta_{21}(t)$

the proposed control strategy.

Simulation model. The linearized lateral dynamics of Boeing 747 can be described as ([43])

$$\dot{x}(t) = Ax(t) + Bu(t), y(t) = Cx(t), \quad (55)$$

where $x(t) = [\beta, r_b, p, \phi]$ is the state vector with β, r_b, p, and ϕ being the side-slip angle, the yaw rate, the roll rate, and the roll angle, respectively, $y(t) = [y_1, y_2]^T = [\beta, r_b]^T$ is the system output vector consisting of the side-slip angle β and the yaw rate r_b, and $u(t) = [u_1, u_2]^T = [\delta_{r1}, \delta_{r2}]^T$ is the control input vector consisting of two rudder servos. From the data provided in [43], in horizontal flight at 40000 ft and nominal forward speed 774 ft/sec (Mach 0.8), the Boeing 747 lateral dynamics matrices are

$$A = \begin{bmatrix} -0.0558 & -0.9968 & 0.0802 & 0.0415 \\ 0.5980 & -0.1150 & -0.0318 & 0.0000 \\ -3.0500 & 0.3880 & -0.4650 & 0.0000 \\ 0.0000 & 0.0805 & 1.0000 & 0.0000 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.0073 & 0.0100 \\ -0.4750 & -0.5000 \\ 0.1530 & 0.2000 \\ 0.0000 & 0.0000 \end{bmatrix}.$$

Using the zero-order holder method and a sampling time $T = 0.1$ sec, we obtain the corresponding discrete-time system state space model of (55) as ([44])

$$x(t+1) = Ax(t) + Bu(t), y(t) = Cx(t), \quad (56)$$

where

$$A = \begin{bmatrix} 0.9902 & -0.0985 & 0.0082 & 0.0041 \\ 0.0597 & 0.9855 & -0.0028 & 0.0001 \\ -0.2956 & 0.0525 & 0.9533 & -0.0006 \\ -0.0147 & 0.0104 & 0.0977 & 1.0000 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.0031 & 0.0036 \\ -0.0472 & -0.0497 \\ 0.0137 & 0.0182 \\ 0.0005 & 0.0007 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \quad (57)$$

From the state space representation (57), we calculate the transfer function of system (56) as $G(z) = C(zI - A)^{-1}B = Z(z)P^{-1}(z)$, where $P(z) = \text{diag}\{z^3 - 2.9832z^2 + 2.9756z - 0.9924, z^3 - 2.9832z^2 + 2.9756z - 0.9924\}$ and $Z(z) = [Z_1(z), Z_2(z)]$ with $Z_1(z) = [0.0031z^2 - 0.0014z - 0.0017, -0.0472z^2 + 0.0936z - 0.0469]^T$ and $Z_2(z) = [0.0036z^2 - 0.0021z - 0.0015, -0.0497z^2 + 0.0986z - 0.0494]^T$. Then, we determine the modified left interactor matrix of $G(z)$ is the same as (53). Thus, we get the high-frequency gain matrix K_p of $G(z)$ as

$$K_p = \lim_{z \to \infty} \xi_m(z)G(z) = \begin{bmatrix} 0.0031 & 0.0036 \\ -0.0472 & -0.0497 \end{bmatrix}. \quad (58)$$

Then, we can verify that the system (56) is minimum phase. Moreover, the observability index of the system (56) is 2. From (58), we calculate $\Delta_1 = 0.0031$ and $\Delta_2 = 0.0000158$ in this simulation, which are non-zero. The reference model is chosen as $y^*(t) = \xi_m(z)[r](t)$ with $r(t) = [0.01 \sin 0.2t \text{ deg}, 0.01 \cos 0.2t \text{ deg/sec}]^T$.

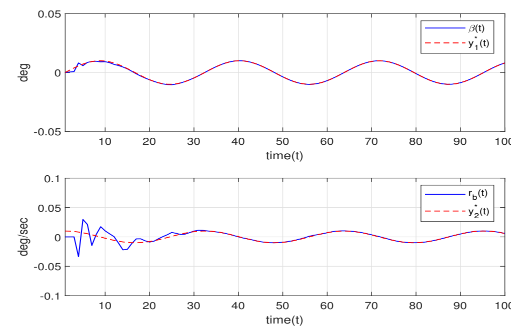

Fig. 6. Trajectories of $\beta(t), r_b(t)$ and the reference output $y^*(t)$

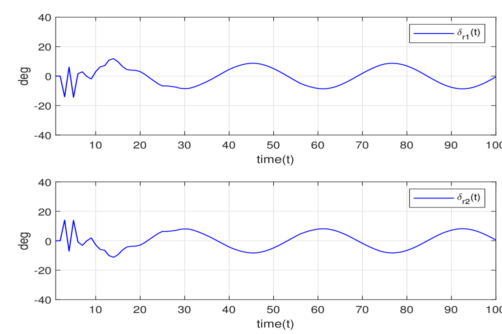

Fig. 7. Trajectories of the control inputs $\delta_{r1}(t), \delta_{r2}(t)$

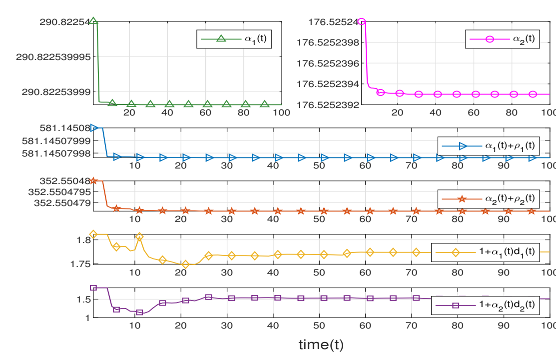

Fig. 8. Trajectories of $\alpha_i(t), \alpha_i(t) + \rho_i(t)$, and $1 + \alpha_i(t)d_i(t)$

Parameter setting. With the observability index being 2, we choose $\lambda(z) = z$ and $A(z) = I_2$ in this simulation. Utilizing the matching equation (4), we determine

$$\Theta_1^{*T} = \begin{bmatrix} -29.4746 & -31.9747 \\ 2.8976 & 3.9428 \end{bmatrix},$$

$$\Theta_2^{*T} = \begin{bmatrix} -3114.3243 & 738.0676 \\ 2957.6681 & -169.7057 \end{bmatrix},$$

$$\Theta_{20}^{*} = \begin{bmatrix} 6221.1230 & -511.7109 \\ -5908.1893 & -25.5855 \end{bmatrix},$$

$$\Theta_3^{*} = \begin{bmatrix} -3135.6467 & -227.1293 \\ 2977.9180 & 195.5836 \end{bmatrix}.$$

Moreover, we decompose K_p in (58) as $K_p = LD^*U$ with

$$L = \begin{bmatrix} 1 & 0 \\ -15.2258 & 1 \end{bmatrix}, D^* = \begin{bmatrix} 0.0031 & 0 \\ 0 & 0.0051 \end{bmatrix},$$
$$U = \begin{bmatrix} 1 & 1.1613 \\ 0 & 1 \end{bmatrix}.$$

Then, we can calculate the parameters $\Phi_1^*, \Phi_2^*, \Phi_{20}^*, \Phi_3^*, \Phi_0^*$, and Θ_0^* with determined parameters $\Theta_1^*, \Theta_2^*, \Theta_{20}^*, \Theta_3^*$, and L, D^*, U. Furthermore, we can obtain all needed controller parameters. In this simulation, the initial parameter estimates are chosen as 85% of the true values. Moreover, we choose adaptive gains $\Gamma_{\beta 1} = 0.7I_{20}, \Gamma_{\beta 2} = 0.7I_{18}, \Gamma_{\theta 2} = 0.7, \Gamma_{\lambda 2} = 0.7$, filter operators $h_1(z) = 1/z, h_2(z) = 1/z$, and initial state $x(0) = [0 \text{ deg}, 0 \text{ deg/sec}, 0 \text{ deg/sec}, 0.1 \text{ deg}]^T$. In addition, we choose $\underline{\alpha}_1 = 0.5$ and $\underline{\alpha}_2 = 0.5$ for the time-varying gain functions $\alpha_1(t)$ and $\alpha_2(t)$, respectively. With all chosen initial parameters, the control law and parameter update law can be derived by (17) and (33), respectively.

Simulation figures. The system response is illustrated in Figs. 6-10. Fig. 6 shows the response of the system output $y(t)$ (the side-slip angle $\beta(t)$ and the yaw rate $r_b(t)$), and the reference output $y^*(t)$, which illustrates that $y(t)$ tracks $y^*(t)$ asymptotically. Fig. 7 presents the trajectories of the control inputs δ_{r1} and δ_{r2}. Fig. 8 shows the trajectories of time-varying gain functions $\alpha_1(t)$ and $\alpha_2(t)$, as well as corresponding singularity-free indices $\alpha_i(t) + \rho_i(t)$ and $1 + \alpha_i(t)d_i(t)$, $i = 1, 2$, for the simulation model (56). It displays that $\alpha_i(t) + \rho_i(t) \neq 0$ and $1 + \alpha_i(t)d_i(t) \neq 0$, $i = 1, 2$, in the adaptive control process. Fig. 9 and Fig. 10 show the trajectories of part of the estimated parameters. Similarly, the parameter estimates may not converge to their nominal values as shown in Fig. 9 and Fig. 10, but the asymptotic tracking control objective is still achieved. In summary, the proposed control method also works for the discrete-time linearized lateral dynamics model (56).

V. Concluding remarks

This paper has addressed the parameter estimation and adaptive singularity-free control problem for a general class of MIMO discrete-time LTI systems with all system parameters being unknown. The development of a novel output feedback adaptive control scheme obviates the necessity for prior knowledge of the sign and bound information of the high-frequency gain matrix. Compared with the existing results, the proposed adaptive control law does not rely on any additional design conditions; however, it still ensures closed-loop stability and asymptotic output tracking without incurring any switching mechanism. Several topics warrants further investigation based on the results of this paper. For instance, exploring the application of the proposed control scheme for systems with time-varying control gain matrices is a topic worth studying. Moreover, extending the proposed method to continuous-time MIMO systems remains unclear, where stability analysis and overall design mechanism are different from discrete-time case of this work. Further, evaluating the robustness of the proposed control scheme under additive/multiplicative external disturbances is crucial for its practical applicability.

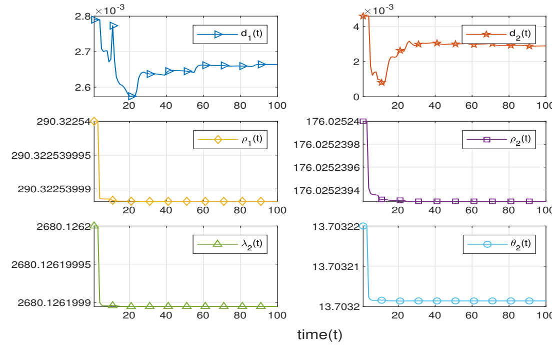

Fig. 9. Trajectories of $d_1(t), d_2(t), \rho_1(t), \rho_2(t), \lambda_2(t), \theta_2(t)$

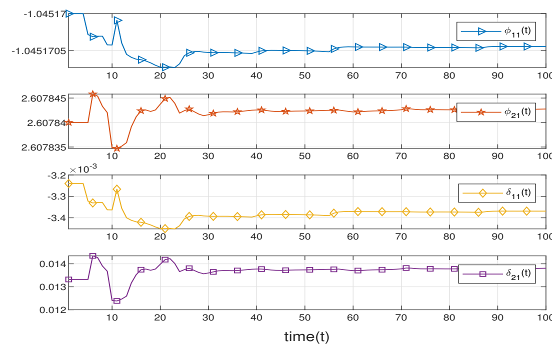

Fig. 10. Trajectories of $\phi_{11}(t), \phi_{21}(t), \delta_{11}(t), \delta_{21}(t)$

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